

ON THE X -RAY TRANSFORM OF PLANAR SYMMETRIC 2-TENSORS

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ABSTRACT. In this paper we study the attenuated X -ray transform of 2-tensors supported in strictly convex bounded subsets in the Euclidean plane. We characterize its range and reconstruct all possible 2-tensors yielding identical X -ray data. The characterization is in terms of a Hilbert-transform associated with A -analytic maps in the sense of Bukhgeim.

1. INTRODUCTION

This paper concerns the range characterization of the attenuated X -ray transform of symmetric 2-tensors in the plane. Range characterization of the non-attenuated X -ray transform of functions (0-tensors) in the Euclidean space has been long known [10, 11, 19], whereas in the case of a constant attenuation some range conditions can be inferred from [17, 1, 2]. For a varying attenuation the two dimensional case has been particularly interesting with inversion formulas requiring new analytical tools: the theory of A -analytic maps originally employed in [3], and ideas from inverse scattering in [24]. Constraints on the range for the two dimensional X -ray transform of functions were given in [25, 4], and a range characterization based on Bukhgeim's theory of A -analytic maps was given in [30].

Inversion of the X -ray transform of higher order tensors has been formulated directly in the setting of Riemmanian manifolds with boundary [32]. The case of 2-tensors appears in the linearization of the boundary rigidity problem. It is easy to see that injectivity can hold only in some restricted class: e.g., the class of solenoidal tensors. For two dimensional simple manifolds with boundary, injectivity with in the solenoidal tensor fields has been establish fairly recent: in the non-attenuated case for 0- and 1-tensors we mention the breakthrough result in [29], and in the attenuated case in [34]; see also [13] for a more general weighted transform. Inversion for the attenuated X -ray transform for solenoidal tensors of rank two and higher can be found in [27], with a range characterization in [28]. In the Euclidean

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case we mention an earlier inversion of the attenuated X -ray transform of solenoidal tensors in [16]; however this work does not address range characterization.

Different from the recent characterization in terms of the scattering relation in [28], in this paper the range conditions are in terms of the Hilbert-transform for A -analytic maps introduced in [30, 31]. Our characterization can be understood as an explicit description of the scattering relation in [26, 27, 28] particularized to the Euclidean setting. In the sufficiency part we reconstruct all possible 2-tensors yielding identical X -ray data; see (30) for the non-attenuated case and (82) for the attenuated case.

For a real symmetric 2-tensor $\mathbf{F} \in L^1(\mathbb{R}^2; \mathbb{R}^{2 \times 2})$,

$$(1) \quad \mathbf{F}(x) = \begin{pmatrix} f_{11}(x) & f_{12}(x) \\ f_{12}(x) & f_{22}(x) \end{pmatrix}, \quad x \in \mathbb{R}^2,$$

and a real valued function $a \in L^1(\mathbb{R}^2)$, the a -attenuated X -ray transform of \mathbf{F} is defined by

$$(2) \quad X_a \mathbf{F}(x, \theta) := \int_{-\infty}^{\infty} \langle \mathbf{F}(x + t\theta) \theta, \theta \rangle \exp \left\{ - \int_t^{\infty} a(x + s\theta) ds \right\} dt,$$

where θ is a direction in the unit sphere \mathbf{S}^1 , and $\langle \cdot, \cdot \rangle$ is the scalar product in \mathbb{R}^2 . For the non attenuated case $a \equiv 0$ we use the notation $X\mathbf{F}$.

In this paper, we consider \mathbf{F} be defined on a strictly convex bounded set $\Omega \subset \mathbb{R}^2$ with vanishing trace at the boundary Γ ; further regularity and the order of vanishing will be specified in the theorems. In the attenuated case we assume $a > 0$ in $\overline{\Omega}$.

For any $(x, \theta) \in \overline{\Omega} \times \mathbf{S}^1$ let $\tau(x, \theta)$ be length of the chord in the direction of θ passing through x . Let also consider the incoming $(-)$, respectively outgoing $(+)$ submanifolds of the unit bundle restricted to the boundary

$$(3) \quad \Gamma_{\pm} := \{(x, \theta) \in \Gamma \times \mathbf{S}^1 : \pm \theta \cdot n(x) > 0\},$$

and the variety

$$(4) \quad \Gamma_0 := \{(x, \theta) \in \Gamma \times \mathbf{S}^1 : \theta \cdot n(x) = 0\},$$

where $n(x)$ denotes outer normal.

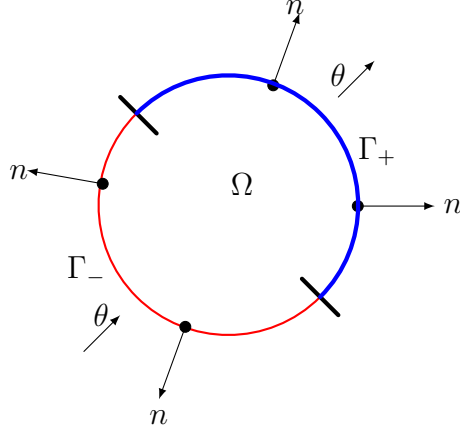
The a -attenuated X -ray transform of \mathbf{F} is realized as a function on Γ_+ by

$$(5) \quad X_a \mathbf{F}(x, \theta) = \int_{-\tau(x, \theta)}^0 \langle \mathbf{F}(x + t\theta) \theta, \theta \rangle e^{-\int_t^0 a(x + s\theta) ds} dt, \quad (x, \theta) \in \Gamma_+.$$

We approach the range characterization through its connection with the transport model as follows: The boundary value problem

$$(6) \quad \theta \cdot \nabla u(x, \theta) + a(x)u(x, \theta) = \langle \mathbf{F}(x)\theta, \theta \rangle \quad (x, \theta) \in \Omega \times \mathbf{S}^1,$$

$$(7) \quad u|_{\Gamma_-} = 0$$

FIGURE 1. Definition of Γ_{\pm}

has a unique solution in $\Omega \times \mathbf{S}^1$ and

$$(8) \quad u|_{\Gamma_+}(x, \theta) = X_a \mathbf{F}(x, \theta), \quad (x, \theta) \in \Gamma_+.$$

The X -ray transform of 2-tensors occurs in the linearization of the boundary rigidity problem [32]: For $\epsilon > 0$ small, let

$$g^\epsilon(x) := \mathbf{I} + \epsilon \mathbf{F}(x) + o(\epsilon), \quad x \in \Omega,$$

be a family of metrics perturbations from the Euclidean, where \mathbf{I} is the identity matrix and \mathbf{F} is as in (1). For an arbitrary pair of boundary points $x, y \in \Gamma$ let $d_\epsilon(x, y)$ denote their distance in the metric g^ϵ . The boundary rigidity problem asks for the recovery of the metric g^ϵ from knowledge of $d_\epsilon(x, y)$ for all $x, y \in \Gamma$. In the linearized case one seeks to recover $\mathbf{F}(x)$ from $\frac{d}{d\epsilon} \Big|_{\epsilon=0} d_\epsilon^2(x, y)$. Taking into account the length minimizing property of geodesic one can show that

$$\frac{1}{|x - y|} \frac{d}{d\epsilon} \Big|_{\epsilon=0} d_\epsilon^2(x, y) = \int_{-|x-y|}^0 \langle \mathbf{F}(x + t\theta)\theta, \theta \rangle dt = X\mathbf{F}(x, \theta),$$

where $\theta := \frac{x - y}{|x - y|} \in \mathbf{S}^1$.

2. PRELIMINARIES

In this section we briefly introduce the properties of Bukhgeim's A -analytic maps [7] needed later.

For $z = x_1 + ix_2$, we consider the Cauchy-Riemann operators

$$(9) \quad \bar{\partial} = (\partial_{x_1} + i\partial_{x_2})/2, \quad \partial = (\partial_{x_1} - i\partial_{x_2})/2.$$

Let $l_\infty(, l_1)$ be the space of bounded (, respectively summable) sequences, $\mathcal{L} : l_\infty \rightarrow l_\infty$ be the left shift

$$\mathcal{L}\langle u_{-1}, u_{-2}, \dots \rangle = \langle u_{-2}, u_{-3}, u_{-4}, \dots \rangle.$$

Definition 2.1. *A sequence valued map*

$$z \mapsto \mathbf{u}(z) := \langle u_{-1}(z), u_{-2}(z), u_{-3}(z), \dots \rangle$$

is called \mathcal{L} -analytic, if $\mathbf{u} \in C(\overline{\Omega}; l_\infty) \cap C^1(\Omega; l_\infty)$ and

$$(10) \quad \overline{\partial} \mathbf{u}(z) + \mathcal{L} \partial \mathbf{u}(z) = 0, \quad z \in \Omega.$$

For $0 < \alpha < 1$ and $k = 1, 2$, we recall the Banach spaces in [30]:

$$(11) \quad l_\infty^{1,k}(\Gamma) := \left\{ \mathbf{u} = \langle u_{-1}, u_{-2}, \dots \rangle : \sup_{\zeta \in \Gamma} \sum_{j=1}^{\infty} j^k |u_{-j}(\zeta)| < \infty \right\},$$

$$(12) \quad C^\alpha(\Gamma; l_1) := \left\{ \mathbf{u} : \sup_{\xi \in \Gamma} \|\mathbf{u}(\xi)\|_{l_1} + \sup_{\substack{\xi, \eta \in \Gamma \\ \xi \neq \eta}} \frac{\|\mathbf{u}(\xi) - \mathbf{u}(\eta)\|_{l_1}}{|\xi - \eta|^\alpha} < \infty \right\}.$$

By replacing Γ with $\overline{\Omega}$ and l_1 with l_∞ in (12) we similarly define $C^\alpha(\overline{\Omega}; l_1)$, respectively, $C^\alpha(\overline{\Omega}; l_\infty)$.

At the heart of the theory of A -analytic maps lies a Cauchy-like integral formula introduced by Bukhgeim in [7]. The explicit variant (13) appeared first in Finch [8]. The formula below is restated in terms of \mathcal{L} -analytic maps as in [31].

Theorem 2.1. [31, Theorem 2.1] *For some $\mathbf{g} = \langle g_{-1}, g_{-2}, g_{-3}, \dots \rangle \in l_\infty^{1,1}(\Gamma) \cap C^\alpha(\Gamma; l_1)$ define the Bukhgeim-Cauchy operator \mathcal{B} acting on \mathbf{g} ,*

$$\Omega \ni z \mapsto \langle (\mathcal{B}\mathbf{g})_{-1}(z), (\mathcal{B}\mathbf{g})_{-2}(z), (\mathcal{B}\mathbf{g})_{-3}(z), \dots \rangle,$$

by

$$(13) \quad \begin{aligned} (\mathcal{B}\mathbf{g})_{-n}(z) := & \frac{1}{2\pi i} \sum_{j=0}^{\infty} \int_{\Gamma} \frac{g_{-n-j}(\zeta) \overline{(\zeta - z)}^j}{(\zeta - z)^{j+1}} d\zeta \\ & - \frac{1}{2\pi i} \sum_{j=1}^{\infty} \int_{\Gamma} \frac{g_{-n-j}(\zeta) \overline{(\zeta - z)}^{j-1}}{(\zeta - z)^j} d\overline{\zeta}, \quad n = 1, 2, 3, \dots \end{aligned}$$

Then $\mathcal{B}\mathbf{g} \in C^{1,\alpha}(\Omega; l_\infty) \cap C(\overline{\Omega}; l_\infty)$ and it is also \mathcal{L} -analytic.

For our purposes further regularity in $\mathcal{B}\mathbf{g}$ will be required. Such smoothness is obtained by increasing the assumptions on the rate of decay of the

terms in \mathbf{g} as explicit below. For $0 < \alpha < 1$, let us recall the Banach space Y_α in [30]:

$$(14) \quad Y_\alpha = \left\{ \mathbf{g} \in l_\infty^{1,2}(\Gamma) : \sup_{\substack{\xi, \mu \in \Gamma \\ \xi \neq \mu}} \sum_{j=1}^{\infty} j \frac{|g_{-j}(\xi) - g_{-j}(\mu)|}{|\xi - \mu|^\alpha} < \infty \right\}.$$

Proposition 2.1. [31, Proposition 2.1] *If $\mathbf{g} \in Y_\alpha$, $\alpha > 1/2$, then*

$$(15) \quad \mathcal{B}\mathbf{g} \in C^{1,\alpha}(\Omega; l_1) \cap C^\alpha(\overline{\Omega}; l_1) \cap C^2(\Omega; l_\infty).$$

The Hilbert transform associated with boundary of \mathcal{L} -analytic maps is defined below.

Definition 2.2. *For $\mathbf{g} = \langle g_{-1}, g_{-2}, g_{-3}, \dots \rangle \in l_\infty^{1,1}(\Gamma) \cap C^\alpha(\Gamma; l_1)$, we define the Hilbert transform $\mathcal{H}\mathbf{g}$ componentwise for $n \geq 1$ by*

$$(16) \quad \begin{aligned} (\mathcal{H}\mathbf{g})_{-n}(\xi) &= \frac{1}{\pi} \int_\Gamma \frac{g_{-n}(\zeta)}{\zeta - \xi} d\zeta \\ &+ \frac{1}{\pi} \int_\Gamma \left\{ \frac{d\zeta}{\zeta - \xi} - \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{\xi}} \right\} \sum_{j=1}^{\infty} g_{-n-j}(\zeta) \left(\frac{\bar{\zeta} - \bar{\xi}}{\zeta - \xi} \right)^j, \quad \xi \in \Gamma. \end{aligned}$$

The following result justifies the name of the transform \mathcal{H} . For its proof we refer to [30, Theorem 3.2].

Theorem 2.2. *For $0 < \alpha < 1$, let $\mathbf{g} \in l_\infty^{1,1}(\Gamma) \cap C^\alpha(\Gamma; l_1)$. For \mathbf{g} to be boundary value of an \mathcal{L} -analytic function it is necessary and sufficient that*

$$(17) \quad (I + i\mathcal{H})\mathbf{g} = \mathbf{0},$$

where \mathcal{H} is as in (16).

3. THE NON-ATTENUATED CASE

In this section we assume $a \equiv 0$. We establish necessary and sufficient conditions for a sufficiently smooth function on $\Gamma \times \mathbf{S}^1$ to be the X -ray data of some sufficiently smooth real valued symmetric 2-tensor \mathbf{F} . For $\theta = (\cos \varphi, \sin \varphi) \in \mathbf{S}^1$, a calculation shows that

$$(18) \quad \langle \mathbf{F}(x)\theta, \theta \rangle = f_0(x) + \overline{f_2(x)}e^{2i\varphi} + f_2(x)e^{-2i\varphi},$$

where

$$(19) \quad f_0(x) = \frac{f_{11}(x) + f_{22}(x)}{2}, \text{ and } f_2(x) = \frac{f_{11}(x) - f_{22}(x)}{4} + i\frac{f_{12}(x)}{2}.$$

The transport equation in (6) becomes

$$(20) \quad \theta \cdot \nabla u(x, \theta) = f_0(x) + \overline{f_2(x)}e^{2i\varphi} + f_2(x)e^{-2i\varphi}, \quad x \in \Omega.$$

For $z = x_1 + ix_2 \in \Omega$, we consider the Fourier expansions of $u(z, \cdot)$ in the angular variable $\theta = (\cos \varphi, \sin \varphi)$:

$$u(z, \theta) = \sum_{n=-\infty}^{\infty} u_n(z) e^{in\varphi}.$$

Since u is real valued its Fourier modes occur in conjugates,

$$u_{-n}(z) = \overline{u_n(z)}, \quad n \geq 0, \quad z \in \Omega.$$

With the Cauchy-Riemann operators defined in (9) the advection operator becomes

$$\theta \cdot \nabla = e^{-i\varphi} \bar{\partial} + e^{i\varphi} \partial.$$

Provided appropriate convergence of the series (given by smoothness in the angular variable) we see that if u solves (20) then its Fourier modes solve the system

$$(21) \quad \bar{\partial} u_1(z) + \partial u_{-1}(z) = f_0(z),$$

$$(22) \quad \bar{\partial} u_{-1}(z) + \partial u_{-3}(z) = f_2(z),$$

$$(23) \quad \bar{\partial} u_{2n}(z) + \partial u_{2n-2}(z) = 0, \quad n \leq 0,$$

$$(24) \quad \bar{\partial} u_{2n-1}(z) + \partial u_{2n-3}(z) = 0, \quad n \leq -1,$$

The range characterization is given in terms of the trace

$$(25) \quad g := u|_{\Gamma \times \mathbb{S}^1} = \begin{cases} X\mathbf{F}(x, \theta), & (x, \theta) \in \Gamma_+, \\ 0, & (x, \theta) \in \Gamma_- \cup \Gamma_0. \end{cases}$$

More precisely, in terms of its Fourier modes in the angular variables:

$$(26) \quad g(\zeta, \theta) = \sum_{n=-\infty}^{\infty} g_n(\zeta) e^{in\varphi}, \quad \zeta \in \Gamma.$$

Since the trace g is also real valued, its Fourier modes will satisfy

$$(27) \quad g_{-n}(\zeta) = \overline{g_n(\zeta)}, \quad n \geq 0, \quad \zeta \in \Gamma.$$

From the negative even modes, we built the sequence

$$(28) \quad \mathbf{g}^{even} := \langle g_0, g_{-2}, g_{-4}, \dots \rangle.$$

From the negative odd modes starting from mode -3 , we built the sequence

$$(29) \quad \mathbf{g}^{odd} := \langle g_{-3}, g_{-5}, g_{-7}, \dots \rangle.$$

Next we characterize the data g in terms of the Hilbert Transform \mathcal{H} in (16). We will construct simultaneously the right hand side of the transport equation (20) and the solution u whose trace matches the boundary data g . Construction of u is via its Fourier modes. We first construct the negative modes and then the positive modes are constructed by conjugation.

Except from negative one mode u_{-1} all non-positive modes are defined by Bukhgeim-Cauchy integral formula in (13) using boundary data. Other then having the trace g_{-1} on the boundary u_{-1} is unconstrained. It is chosen arbitrarily from the class of functions

$$(30) \quad \Psi_g := \left\{ \psi \in C^1(\overline{\Omega}; \mathbb{C}) : \psi|_{\Gamma} = g_{-1} \right\}.$$

Theorem 3.1 (Range characterization in the non-attenuated case). *Let $\alpha > 1/2$.*

(i) *Let $\mathbf{F} \in C_0^{1,\alpha}(\Omega; \mathbb{R}^{2 \times 2})$. For $g := \begin{cases} X\mathbf{F}(x, \theta), & (x, \theta) \in \Gamma_+, \\ 0, & (x, \theta) \in \Gamma_- \cup \Gamma_0, \end{cases}$ consider the corresponding sequences \mathbf{g}^{even} as in (28) and \mathbf{g}^{odd} as in (29). Then $\mathbf{g}^{even}, \mathbf{g}^{odd} \in l_\infty^{1,1}(\Gamma) \cap C^\alpha(\Gamma; l_1)$ satisfy*

$$(31) \quad [I + i\mathcal{H}]\mathbf{g}^{even} = \mathbf{0},$$

$$(32) \quad [I + i\mathcal{H}]\mathbf{g}^{odd} = \mathbf{0},$$

where the operator \mathcal{H} is the Hilbert transform in (16).

(ii) *Let $g \in C^\alpha(\Gamma; C^{1,\alpha}(\mathbf{S}^1)) \cap C(\Gamma; C^{2,\alpha}(\mathbf{S}^1))$ be real valued with $g|_{\Gamma_- \cup \Gamma_0} = 0$. If the corresponding sequence $\mathbf{g}^{even}, \mathbf{g}^{odd} \in Y_\alpha$ satisfies (31) and (32), then there exists a real valued symmetric 2-tensor $\mathbf{F} \in C(\Omega; \mathbb{R}^{2 \times 2})$, such that $g|_{\Gamma_+} = X\mathbf{F}$. Moreover for each $\psi \in \Psi_g$ in (30), there is a unique real valued symmetric 2-tensor \mathbf{F}_ψ such that $g|_{\Gamma_+} = X\mathbf{F}_\psi$.*

Proof. (i) **Necessity**

Let $\mathbf{F} \in C_0^{1,\alpha}(\Omega; \mathbb{R}^{2 \times 2})$. Since \mathbf{F} is compactly supported inside Ω , for any point at the boundary there is a cone of lines which do not meet the support. Thus $g \equiv 0$ in the neighborhood of the variety Γ_0 which yields $g \in C^{1,\alpha}(\Gamma \times \mathbf{S}^1)$. Moreover, g is the trace on $\Gamma \times \mathbf{S}^1$ of a solution $u \in C^{1,\alpha}(\overline{\Omega} \times \mathbf{S}^1)$ of the transport equation (20). By [30, Proposition 4.1] $\mathbf{g}^{even}, \mathbf{g}^{odd} \in l_\infty^{1,1}(\Gamma) \cap C^\alpha(\Gamma; l_1)$.

If u solves (20) then its Fourier modes satisfy (21), (22), (23) and (24). Since the negative even Fourier modes u_{2n} of u satisfies the system (23) for $n \leq 0$, then

$$z \mapsto \mathbf{u}^{even}(z) := \langle u_0(z), u_{-2}(z), u_{-4}(z), u_{-6}(z), \dots \rangle$$

is \mathcal{L} -analytic in Ω and the necessity part in Theorem 2.2 yields (31).

The equation (24) for negative odd Fourier modes u_{2n-1} starting from mode -3 yield that the sequence valued map

$$z \mapsto \mathbf{u}^{odd}(z) := \langle u_{-3}(z), u_{-5}(z), u_{-7}(z), \dots \rangle$$

is \mathcal{L} -analytic in Ω and the necessity part in Theorem 2.2 yields (32).

(ii) **Sufficiency**

To prove the sufficiency we will construct a real valued symmetric 2-tensor \mathbf{F} in Ω and a real valued function $u \in C^1(\Omega \times \mathbf{S}^1) \cap C(\bar{\Omega} \times \mathbf{S}^1)$ such that $u|_{\Gamma \times \mathbf{S}^1} = g$ and u solves (20) in Ω . The construction of such u is in terms of its Fourier modes in the angular variable and it is done in several steps.

Step 1: The construction of negative even modes u_{2n} for $n \leq 0$.

Let $g \in C^\alpha(\Gamma; C^{1,\alpha}(\mathbf{S}^1)) \cap C(\Gamma; C^{2,\alpha}(\mathbf{S}^1))$ be real valued with $g|_{\Gamma_- \cup \Gamma_0} = 0$. Let the corresponding sequences \mathbf{g}^{even} satisfying (31) and \mathbf{g}^{odd} satisfying (32). By [30, Proposition 4.1(ii)] $\mathbf{g}^{even}, \mathbf{g}^{odd} \in Y_\alpha$. Use the Bukhgeim-Cauchy Integral formula (13) to construct the negative even Fourier modes:

$$(33) \quad \langle u_0(z), u_{-2}(z), u_{-4}(z), u_{-6}(z), \dots \rangle := \mathcal{B}\mathbf{g}^{even}(z), \quad z \in \Omega.$$

By Theorem 2.1, the sequence valued map

$$z \mapsto \langle u_0(z), u_{-2}(z), u_{-4}(z), \dots \rangle,$$

is \mathcal{L} -analytic in Ω , thus the equations

$$(34) \quad \bar{\partial}u_{-2k} + \partial u_{-2k-2} = 0,$$

are satisfied for all $k \geq 0$. Moreover, the hypothesis (31) and the sufficiency part of Theorem 2.2 yields that they extend continuously to Γ and

$$(35) \quad u_{-2k}|_\Gamma = g_{-2k}, \quad k \geq 0.$$

Step 2: The construction of positive even modes u_{2n} for $n \geq 1$.

All of the positive even Fourier modes are constructed by conjugation:

$$(36) \quad u_{2k} := \overline{u_{-2k}}, \quad k \geq 1.$$

By conjugating (34) we note that the positive even Fourier modes also satisfy

$$(37) \quad \bar{\partial}u_{2k+2} + \partial u_{2k} = 0, \quad k \geq 0.$$

Moreover, they extend continuously to Γ and

$$(38) \quad u_{2k}|_\Gamma = \overline{u_{-2k}|_\Gamma} = \overline{g_{-2k}} = g_{2k}, \quad k \geq 1.$$

Thus, as a summary, we have shown that

$$(39) \quad \bar{\partial}u_{2k} + \partial u_{2k-2} = 0, \quad \forall k \in \mathbb{Z},$$

$$(40) \quad u_{2k}|_\Gamma = g_{2k}, \quad \forall k \in \mathbb{Z}.$$

Step 3: The construction of modes u_{-1} and u_1 .

Let $\psi \in \Psi_g$ as in (30). We define

$$(41) \quad u_{-1} := \psi, \quad \text{and} \quad u_1 := \overline{\psi}.$$

Since g is real valued, we have

$$(42) \quad u_1|_\Gamma = \overline{g_{-1}} = g_1.$$

Step 4: The construction of negative odd modes u_{2n-1} for $n \leq -1$.

Use the Bukhgeim-Cauchy Integral formula (13) to construct the other odd negative Fourier modes:

$$(43) \quad \langle u_{-3}(z), u_{-5}(z), \dots \rangle := \mathcal{B}\mathbf{g}^{odd}(z), \quad z \in \Omega.$$

By Theorem 2.1, the sequence valued map

$$z \mapsto \langle u_{-3}(z), u_{-5}(z), u_{-7}(z), \dots \rangle,$$

is \mathcal{L} -analytic in Ω , thus the equations

$$(44) \quad \bar{\partial}u_{2k-1} + \partial u_{2k-3} = 0,$$

are satisfied for all $k \leq -1$. Moreover, the hypothesis (32) and the sufficiency part of Theorem 2.2 yields that they extend continuously to Γ and

$$(45) \quad u_{2k-1}|_{\Gamma} = g_{2k-1}, \quad \forall k \leq -1.$$

Step 5: The construction of positive odd modes u_{2n+1} for $n \geq 1$.

All of the positive odd Fourier modes are constructed by conjugation:

$$(46) \quad u_{2k+3} := \overline{u_{-(2k+3)}}, \quad k \geq 0.$$

By conjugating (44) we note that the positive odd Fourier modes also satisfy

$$(47) \quad \bar{\partial}u_{2k+3} + \partial u_{2k+1} = 0, \quad \forall k \geq 1.$$

Moreover, they extend continuously to Γ and

$$(48) \quad u_{2k+3}|_{\Gamma} = \overline{u_{-(2k+3)}|_{\Gamma}} = \overline{g_{-(2k+3)}} = g_{2k+3}, \quad k \geq 0.$$

Step 6: The construction of the tensor field \mathbf{F}_{ψ} whose X-ray data is g .

We define the 2-tensor field

$$(49) \quad \mathbf{F}_{\psi} := \begin{pmatrix} f_0 + 2\operatorname{Re} f_2 & 2\operatorname{Im} f_2 \\ 2\operatorname{Im} f_2 & f_0 - 2\operatorname{Re} f_2 \end{pmatrix},$$

where

$$(50) \quad f_0 = 2\operatorname{Re}(\partial\psi), \text{ and } f_2 = \bar{\partial}\psi + \partial u_{-3}.$$

In order to show $g|_{\Gamma_+} = X\mathbf{F}_{\psi}$ with \mathbf{F}_{ψ} as in (49), we define the real valued function u via its Fourier modes

$$(51) \quad u(z, \theta) := u_0(z) + \psi(z)e^{-i\varphi} + \bar{\psi}(z)e^{i\varphi} + \sum_{n=2}^{\infty} u_{-n}(z)e^{-in\varphi} + \sum_{n=2}^{\infty} u_n(z)e^{in\varphi},$$

and check that it has the trace g on Γ and satisfies the transport equation (20).

Since $g \in C^{\alpha}(\Gamma; C^{1,\alpha}(\mathbf{S}^1)) \cap C(\Gamma; C^{2,\alpha}(\mathbf{S}^1))$, we use [30, Corollary 4.1] and [30, Proposition 4.1 (iii)] to conclude that u defined in (51) belongs to

$C^{1,\alpha}(\Omega \times \mathbf{S}^1) \cap C^\alpha(\overline{\Omega} \times \mathbf{S}^1)$. In particular $u(\cdot, \theta)$ for $\theta = (\cos \varphi, \sin \varphi)$ extends to the boundary and its trace satisfies

$$\begin{aligned} u(\cdot, \theta)|_\Gamma &= \left(u_0 + \psi e^{-i\varphi} + \overline{\psi} e^{i\varphi} + \sum_{n=2}^{\infty} u_{-n} e^{-in\varphi} + \sum_{n=2}^{\infty} u_n e^{in\varphi} \right) \Big|_\Gamma \\ &= u_0|_\Gamma + \psi|_\Gamma e^{-i\varphi} + \overline{\psi}|_\Gamma e^{i\varphi} + \sum_{n=2}^{\infty} u_{-n}|_\Gamma e^{-in\varphi} + \sum_{n=2}^{\infty} u_n|_\Gamma e^{in\varphi} \\ &= g_0 + g_{-1} e^{-i\varphi} + g_1 e^{i\varphi} + \sum_{n=2}^{\infty} g_{-n} e^{-in\varphi} + \sum_{n=2}^{\infty} g_n e^{in\varphi} \\ &= g(\cdot, \theta), \end{aligned}$$

where in the third equality above we used (40), (45), (48), (42) and definition of $\psi \in \Psi_g$ in (30).

Since $u \in C^{1,\alpha}(\Omega \times \mathbf{S}^1) \cap C^\alpha(\overline{\Omega} \times \mathbf{S}^1)$, the following calculation is also justified:

$$\begin{aligned} \theta \cdot \nabla u &= e^{-i\varphi} \overline{\partial} u_0 + e^{i\varphi} \partial u_0 + e^{-2i\varphi} \overline{\partial} \psi + \overline{\partial} \overline{\psi} + \partial \psi + e^{2i\varphi} \partial \overline{\psi} \\ &\quad + \sum_{n=2}^{\infty} \overline{\partial} u_{-n} e^{-i(n+1)\varphi} + \sum_{n=2}^{\infty} \partial u_{-n} e^{-i(n-1)\varphi} \\ &\quad + \sum_{n=2}^{\infty} \overline{\partial} u_n e^{i(n-1)\varphi} + \sum_{n=2}^{\infty} \partial u_n e^{i(n+1)\varphi}. \end{aligned}$$

Rearranging the modes in the above equation yields

$$\begin{aligned} \theta \cdot \nabla u &= e^{-2i\varphi} (\overline{\partial} \psi + \partial u_{-3}) + e^{2i\varphi} (\partial \overline{\psi} + \overline{\partial} u_3) + \overline{\partial} \overline{\psi} + \partial \psi \\ &\quad + e^{-i\varphi} (\overline{\partial} u_0 + \partial u_{-2}) + e^{i\varphi} (\partial u_0 + \overline{\partial} u_2) \\ &\quad + \sum_{n=1}^{\infty} (\overline{\partial} u_{-n} + \partial u_{-n-2}) e^{-i(n+1)\varphi} + \sum_{n=1}^{\infty} (\overline{\partial} u_{n+2} + \partial u_n) e^{i(n+1)\varphi}. \end{aligned}$$

Using (39), (44), and (47) simplifies the above equation

$$\theta \cdot \nabla u = e^{-2i\varphi} (\overline{\partial} \psi + \partial u_{-3}) + e^{2i\varphi} (\partial \overline{\psi} + \overline{\partial} u_3) + \overline{\partial} \overline{\psi} + \partial \psi.$$

Now using (50), we conclude (20).

$$\theta \cdot \nabla u = e^{-2i\varphi} f_2 + e^{2i\varphi} \overline{f_2} + f_0 = \langle \mathbf{F}_\psi \theta, \theta \rangle.$$

□

As the source is supported inside, there are no incoming fluxes: hence the trace of a solution u of (20) on Γ_- is zero. We give next a range condition

only in terms of g on Γ_+ , where $g := u|_{\Gamma \times \mathbf{S}^1}$. More precisely, let \tilde{u} be the solution of the boundary value problem

$$(52) \quad \begin{aligned} \theta \cdot \nabla \tilde{u}(x, \theta) &= \langle \mathbf{F}(x)\theta, \theta \rangle, \quad x \in \Omega, \\ \tilde{u}(z, \theta) &= -\frac{1}{2}g|_{\Gamma_+}(z, -\theta), \quad (z, \theta) \in \Gamma_-. \end{aligned}$$

Then one can see that

$$(53) \quad \tilde{u}|_{\Gamma_+} = \frac{1}{2}g|_{\Gamma_+},$$

and therefore $\tilde{u}|_{\Gamma \times \mathbf{S}^1}$ is an odd function of θ . This shows that we can work with the following odd extension:

$$(54) \quad \tilde{g}(z, \theta) := \frac{g(z, \theta) - g(z, -\theta)}{2}, \quad (z, \theta) \in (\Gamma \times \mathbf{S}^1) \setminus \Gamma_0,$$

and $\tilde{g} = 0$ on Γ_0 . Note that \tilde{g} is the trace of \tilde{u} on $\Gamma \times \mathbf{S}^1$.

The range characterization can be given now in terms of the odd Fourier modes of \tilde{g} , namely in terms of

$$(55) \quad \tilde{\mathbf{g}} := \langle \tilde{g}_{-3}, \tilde{g}_{-5}, \tilde{g}_{-7}, \dots \rangle.$$

Corollary 3.1. *Let $\alpha > 1/2$.*

(i) Let $\mathbf{F} \in C_0^{1,\alpha}(\Omega; \mathbb{R}^{2 \times 2})$, \tilde{u} be the solution of (52) and $\tilde{\mathbf{g}}$ as in (55). Then $\tilde{\mathbf{g}} \in l_\infty^{1,1}(\Gamma) \cap C^\alpha(\Gamma; l_1)$ and

$$(56) \quad [I + i\mathcal{H}]\tilde{\mathbf{g}} = 0,$$

where the operator \mathcal{H} is the Hilbert transform in (16).

(ii) Let $g \in C^\alpha(\Gamma; C^{1,\alpha}(\mathbf{S}^1)) \cap C(\Gamma; C^{2,\alpha}(\mathbf{S}^1))$ be real valued with $g|_{\Gamma_- \cup \Gamma_0} = 0$. Let \tilde{g} be its odd extension as in (54) and the corresponding $\tilde{\mathbf{g}}$ as in (55). If $\tilde{\mathbf{g}}$ satisfies (56), then there exists a real valued symmetric 2-tensor $\mathbf{F} \in C(\Omega; \mathbb{R}^{2 \times 2})$, such that $g|_{\Gamma_+} = X\mathbf{F}$. Moreover for each $\psi \in \Psi_g$ in (30), there is a unique real valued symmetric 2-tensor \mathbf{F}_ψ such that $g|_{\Gamma_+} = X\mathbf{F}_\psi$.

4. THE ATTENUATED CASE

In this section we assume an attenuation $a \in C^{2,\alpha}(\overline{\Omega})$, $\alpha > 1/2$ with

$$\min_{\overline{\Omega}} a > 0.$$

We establish necessary and sufficient conditions for a sufficiently smooth function g on $\Gamma \times \mathbf{S}^1$ to be the attenuated X -ray data, with attenuation a , of some sufficiently smooth real symmetric 2-tensor, i.e. g is the trace on $\Gamma \times \mathbf{S}^1$ of some solution u of

$$(57) \quad \theta \cdot \nabla u(x, \theta) + a(x)u(x, \theta) = \langle \mathbf{F}(x)\theta, \theta \rangle, \quad (x, \theta) \in \Gamma \times \mathbf{S}^1.$$

Different from 1-tensor case in [31] (where there is uniqueness), in the 2-tensor case there is non-uniqueness: see the class of function in (82).

As in [30] we start by the reduction to the non-attenuated case via the special integrating factor e^{-h} , where h is explicitly defined in terms of a by

$$(58) \quad h(z, \theta) := Da(z, \theta) - \frac{1}{2} (I - iH) Ra(z \cdot \theta^\perp, \theta),$$

where θ^\perp is orthogonal to θ , $Da(z, \theta) = \int_0^\infty a(z + t\theta) dt$ is the divergence beam transform of the attenuation a , $Ra(s, \theta) = \int_{-\infty}^\infty a(s\theta^\perp + t\theta) dt$ is the Radon transform of the attenuation a , and the classical Hilbert transform $Hh(s) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{h(t)}{s - t} dt$ is taken in the first variable and evaluated at $s = z \cdot \theta^\perp$. The function h was first considered in the work of Natterer [21]; see also [8], and [6] for elegant arguments that show how h extends from \mathbf{S}^1 inside the disk as an analytic map.

The lemma 4.1 and lemma 4.2 below were proven in [31] for a vanishing at the boundary, $a \in C_0^{1,\alpha}(\overline{\Omega})$, $\alpha > 1/2$. We explain here why the vanishing assumption is not necessary: we extend a in a neighbourhood $\tilde{\Omega}$ of Ω with compact support, $\tilde{a} \in C_0^{1,\alpha}(\tilde{\Omega})$. We apply the results [31, Lemma 4.1 and Lemma 4.2] for the extension \tilde{a} and use it on $\overline{\Omega}$.

Lemma 4.1. [31, Lemma 4.1] *Assume $a \in C^{p,\alpha}(\overline{\Omega})$, $p = 1, 2$, $\alpha > 1/2$, and h defined in (58). Then $h \in C^{p,\alpha}(\overline{\Omega} \times \mathbf{S}^1)$ and the following hold*

(i) *h satisfies*

$$(59) \quad \theta \cdot \nabla h(z, \theta) = -a(z), \quad (z, \theta) \in \Omega \times \mathbf{S}^1.$$

(ii) *h has vanishing negative Fourier modes yielding the expansions*

$$(60) \quad e^{-h(z,\theta)} := \sum_{k=0}^{\infty} \alpha_k(z) e^{ik\varphi}, \quad e^{h(z,\theta)} := \sum_{k=0}^{\infty} \beta_k(z) e^{ik\varphi}, \quad (z, \theta) \in \overline{\Omega} \times \mathbf{S}^1,$$

with

(iii)

$$(61) \quad z \mapsto \langle \alpha_1(z), \alpha_2(z), \alpha_3(z), \dots \rangle \in C^{p,\alpha}(\Omega; l_1) \cap C(\overline{\Omega}; l_1),$$

$$(62) \quad z \mapsto \langle \beta_1(z), \beta_2(z), \beta_3(z), \dots \rangle \in C^{p,\alpha}(\Omega; l_1) \cap C(\overline{\Omega}; l_1).$$

(iv) For any $z \in \Omega$

$$(63) \quad \bar{\partial}\beta_0(z) = 0,$$

$$(64) \quad \bar{\partial}\beta_1(z) = -a(z)\beta_0(z),$$

$$(65) \quad \bar{\partial}\beta_{k+2}(z) + \partial\beta_k(z) + a(z)\beta_{k+1}(z) = 0, \quad k \geq 0.$$

(v) For any $z \in \Omega$

$$(66) \quad \bar{\partial}\alpha_0(z) = 0,$$

$$(67) \quad \bar{\partial}\alpha_1(z) = a(z)\alpha_0(z),$$

$$(68) \quad \bar{\partial}\alpha_{k+2}(z) + \partial\alpha_k(z) + a(z)\alpha_{k+1}(z) = 0, \quad k \geq 0.$$

(vi) The Fourier modes $\alpha_k, \beta_k, k \geq 0$ satisfy

$$(69) \quad \alpha_0\beta_0 = 1, \quad \sum_{m=0}^k \alpha_m\beta_{k-m} = 0, \quad k \geq 1.$$

From (59) it is easy to see that u solves (57) if and only if $v := e^{-h}u$ solves

$$(70) \quad \theta \cdot \nabla v(z, \theta) = \langle F(z)\theta, \theta \rangle e^{-h(z, \theta)}.$$

If $u(z, \theta) = \sum_{n=-\infty}^{\infty} u_n(z)e^{in\varphi}$ solves (57), then its Fourier modes satisfy

$$(71) \quad \bar{\partial}u_1(z) + \partial u_{-1}(z) + a(z)u_0(z) = f_0(z),$$

$$(72) \quad \bar{\partial}u_0(z) + \partial u_{-2}(z) + a(z)u_{-1}(z) = 0,$$

$$(73) \quad \bar{\partial}u_{-1}(z) + \partial u_{-3}(z) + a(z)u_{-2}(z) = f_2(z),$$

$$(74) \quad \bar{\partial}u_n(z) + \partial u_{n-2}(z) + a(z)u_{n-1}(z) = 0, \quad n \leq -2,$$

where f_0, f_2 as defined in (19).

Also, if $v := e^{-h}u = \sum_{n=-\infty}^{\infty} v_n(z)e^{in\varphi}$ solves (70), then its Fourier modes satisfy

$$\bar{\partial}v_1(z) + \partial v_{-1}(z) = \alpha_0(z)f_0(z) + \alpha_2(z)f_2(z),$$

$$\bar{\partial}v_0(z) + \partial v_{-2}(z) = \alpha_1(z)f_2(z),$$

$$\bar{\partial}v_{-1}(z) + \partial v_{-3}(z) = \alpha_0(z)f_2(z),$$

$$(75) \quad \bar{\partial}v_n(z) + \partial v_{n-2}(z) = 0, \quad n \leq -2,$$

where α_0, α_1 and α_2 are the Fourier modes in (60), and f_0, f_2 as defined in (19).

The following result shows that the equivalence between (74) and (75) is intrinsic to negative Fourier modes only.

Lemma 4.2. [31, Lemma 4.2] Assume $a \in C^{1,\alpha}(\overline{\Omega})$, $\alpha > 1/2$.

(i) Let $\mathbf{v} = \langle v_{-2}, v_{-3}, \dots \rangle \in C^1(\Omega, l_1)$ satisfy (75), and $\mathbf{u} = \langle u_{-2}, u_{-3}, \dots \rangle$ be defined componentwise by the convolution

$$(76) \quad u_n := \sum_{j=0}^{\infty} \beta_j v_{n-j}, \quad n \leq -2,$$

where β_j 's are the Fourier modes in (60). Then \mathbf{u} solves (74) in Ω .

(ii) Conversely, let $\mathbf{u} = \langle u_{-2}, u_{-3}, \dots \rangle \in C^1(\Omega, l_1)$ satisfy (74), and $\mathbf{v} = \langle v_{-2}, v_{-3}, \dots \rangle$ be defined componentwise by the convolution

$$(77) \quad v_n := \sum_{j=0}^{\infty} \alpha_j u_{n-j}, \quad n \leq -2,$$

where α_j 's are the Fourier modes in (60). Then \mathbf{v} solves (75) in Ω .

The operators $\partial, \bar{\partial}$ in (9) can be rewritten in terms of the derivative in tangential direction ∂_τ and derivative in normal direction ∂_n ,

$$\begin{aligned} \partial_n &= \cos \eta \partial_{x_1} + \sin \eta \partial_{x_2}, \\ \partial_\tau &= -\sin \eta \partial_{x_1} + \cos \eta \partial_{x_2}, \end{aligned}$$

where η is the angle made by the normal to the boundary with x_1 direction (Since the boundary Γ is known, η is a known function on the boundary). In these coordinates

$$(78) \quad \partial = \frac{e^{-i\eta}}{2}(\partial_n - i\partial_\tau), \quad \bar{\partial} = \frac{e^{i\eta}}{2}(\partial_n + i\partial_\tau).$$

Next we characterize the attenuated X-ray data g in terms of its Fourier modes g_0, g_{-1} and the negative index modes $\gamma_{-2}, \gamma_{-3}, \gamma_{-4} \dots$ of

$$(79) \quad e^{-h(\zeta, \theta)} g(\zeta, \theta) = \sum_{k=-\infty}^{\infty} \gamma_k(\zeta) e^{ik\varphi}, \quad \zeta \in \Gamma.$$

To simplify the statement, let

$$(80) \quad \mathbf{g}_h := \langle \gamma_{-2}, \gamma_{-3}, \gamma_{-4} \dots \rangle,$$

and from the negative even, respectively, negative odd Fourier modes, we build the sequences

$$(81) \quad \mathbf{g}_h^{even} = \langle \gamma_{-2}, \gamma_{-4}, \dots \rangle, \quad \text{and} \quad \mathbf{g}_h^{odd} = \langle \gamma_{-3}, \gamma_{-5}, \dots \rangle.$$

Note that γ_{-1} is not included in the \mathbf{g}_h^{odd} definition. As before we construct simultaneously the right hand side of the transport equation (57) together with the solution u . Construction of u is via its Fourier modes. We first construct the negative modes and then the positive modes are constructed by conjugation. Apart from zeroth mode u_0 and negative one mode u_{-1} , all Fourier modes are constructed uniquely from the data $\mathbf{g}_h^{even}, \mathbf{g}_h^{odd}$. The

mode u_0 will be chosen arbitrarily from the class Ψ_g^a with prescribed trace and gradient on the boundary Γ defined as

$$(82) \quad \Psi_g^a := \left\{ \psi \in C^2(\overline{\Omega}; \mathbb{R}) : \psi|_{\Gamma} = g_0, \right. \\ \left. \partial_n \psi|_{\Gamma} = -2 \operatorname{Re} e^{-in} \left(\partial \sum_{j=0}^{\infty} \beta_j (\mathcal{B} \mathbf{g}_h)_{-2-j} \Big|_{\Gamma} + a|_{\Gamma} g_{-1} \right) \right\},$$

where \mathcal{B} be the Bukhgeim-Cauchy operator in (13), β_j 's are the Fourier modes in (60) and \mathbf{g}_h in (80). The mode u_{-1} is define in terms of u_0 , see (99).

Recall the Hilbert transform \mathcal{H} in (16).

Theorem 4.1 (Range characterization in the attenuated case). *Let $a \in C^{2,\alpha}(\overline{\Omega})$, $\alpha > 1/2$ with $\min_{\overline{\Omega}} a > 0$.*

(i) *Let $\mathbf{F} \in C_0^{1,\alpha}(\Omega; \mathbb{R}^{2 \times 2})$. For $g := \begin{cases} X_a \mathbf{F}(x, \theta), & (x, \theta) \in \Gamma_+, \\ 0, & (x, \theta) \in \Gamma_- \cup \Gamma_0, \end{cases}$ consider the corresponding sequences $\mathbf{g}_h^{\text{even}}, \mathbf{g}_h^{\text{odd}}$ as in (81). Then $\mathbf{g}_h^{\text{even}}, \mathbf{g}_h^{\text{odd}} \in l_{\infty}^{1,1}(\Gamma) \cap C^{\alpha}(\Gamma; l_1)$ satisfy*

$$(83) \quad [I + i\mathcal{H}]\mathbf{g}_h^{\text{even}} = 0, \quad [I + i\mathcal{H}]\mathbf{g}_h^{\text{odd}} = 0, \quad \text{and}$$

$$(84) \quad \partial_{\tau} g_0 = -2 \operatorname{Im} e^{-in} \left(\partial \sum_{j=0}^{\infty} \beta_j (\mathcal{B} \mathbf{g}_h)_{-2-j} \Big|_{\Gamma} + a|_{\Gamma} g_{-1} \right),$$

where \mathcal{H} is the Hilbert transform in (16), \mathcal{B} is the Bukhgeim-Cauchy operator in (13), β_j 's are the Fourier modes in (60) and \mathbf{g}_h in (80).

(ii) *Let $g \in C^{\alpha}(\Gamma; C^{1,\alpha}(\mathbf{S}^1)) \cap C(\Gamma; C^{2,\alpha}(\mathbf{S}^1))$ be real valued with $g|_{\Gamma_- \cup \Gamma_0} = 0$. If the corresponding sequences $\mathbf{g}_h^{\text{even}}, \mathbf{g}_h^{\text{odd}} \in Y_{\alpha}$ satisfying (83) and (84) then there exists a symmetric 2-tensor $\mathbf{F} \in C(\Omega; \mathbb{R}^{2 \times 2})$, such that $g|_{\Gamma_+} = X_a \mathbf{F}$. Moreover for each $\psi \in \Psi_g^a$ in (82), there is a unique real valued symmetric 2-tensor \mathbf{F}_{ψ} such that $g|_{\Gamma_+} = X_a \mathbf{F}_{\psi}$.*

Proof. (i) **Necessity**

Let $\mathbf{F} \in C_0^{1,\alpha}(\Omega; \mathbb{R}^{2 \times 2})$. Since \mathbf{F} is compactly supported inside Ω , for any point at the boundary there is a cone of lines which do not meet the support. Thus $g \equiv 0$ in the neighborhood of the variety Γ_0 which yields $g \in C^{1,\alpha}(\Gamma \times \mathbf{S}^1)$. Moreover, g is the trace on $\Gamma \times \mathbf{S}^1$ of a solution $u \in C^{1,\alpha}(\overline{\Omega} \times \mathbf{S}^1)$. By [30, Proposition 4.1] $\mathbf{g}_h^{\text{even}}, \mathbf{g}_h^{\text{odd}} \in l_{\infty}^{1,1}(\Gamma) \cap C^{\alpha}(\Gamma; l_1)$.

Let $v := e^{-h} u = \sum_{n=-\infty}^{\infty} v_n(z) e^{in\varphi}$, then the negative Fourier modes of v satisfy (75). In particular its negative odd subsequence $\langle v_{-3}, v_{-5}, \dots \rangle$ and negative even subsequence $\langle v_{-2}, v_{-4}, \dots \rangle$ are \mathcal{L} -analytic with traces $\mathbf{g}_h^{\text{odd}}$

respectively \mathbf{g}_h^{even} . The necessity part of Theorem 2.2 yields (83):

$$[I + i\mathcal{H}]\mathbf{g}_h^{odd} = 0, \quad [I + i\mathcal{H}]\mathbf{g}_h^{even} = 0.$$

If u solves (57), then its Fourier modes satisfy (71), (72), (73), and (74). The negative Fourier modes of u and v are related by

$$(85) \quad u_n = \sum_{j=0}^{\infty} \beta_j v_{n-j}, \quad n \leq 0,$$

where β_j 's are the Fourier modes in (60). The restriction of (72) to the boundary yields

$$\bar{\partial}u_0|_{\Gamma} = -\partial u_{-2}|_{\Gamma} - (au_{-1})|_{\Gamma}.$$

Expressing $\bar{\partial}$ in the above equation in terms of ∂_{τ} and ∂_n as in (78) yields

$$\frac{e^{i\eta}}{2}(\partial_n + i\partial_{\tau})u_0|_{\Gamma} = -\partial u_{-2}|_{\Gamma} - a|_{\Gamma}g_{-1}.$$

Simplifying the above expression and using $\partial_{\tau}u_0|_{\Gamma} = \partial_{\tau}g_0$, yields

$$\partial_n u_0|_{\Gamma} + i\partial_{\tau}g_0 = -2e^{-i\eta}(\partial u_{-2}|_{\Gamma} + a|_{\Gamma}g_{-1}).$$

The imaginary part of the above equation yields (84). This proves part (i) of the theorem.

(ii) Sufficiency

To prove the sufficiency we will construct a real valued symmetric 2-tensor \mathbf{F} in Ω and a real valued function $u \in C^1(\Omega \times \mathbf{S}^1) \cap C(\bar{\Omega} \times \mathbf{S}^1)$ such that $u|_{\Gamma \times \mathbf{S}^1} = g$ and u solves (57) in Ω . The construction of such u is in terms of its Fourier modes in the angular variable and it is done in several steps.

Step 1: The construction of negative modes u_n for $n \leq -2$.

Let $g \in C^{\alpha}(\Gamma; C^{1,\alpha}(\mathbf{S}^1)) \cap C(\Gamma; C^{2,\alpha}(\mathbf{S}^1))$ be real valued with $g|_{\Gamma \cup \Gamma_0} = 0$. Let the corresponding sequences $\mathbf{g}_h^{even}, \mathbf{g}_h^{odd}$ as in (81) satisfying (83) and (84). By [30, Proposition 4.1(ii)] and [30, Proposition 5.2(iii)] $\mathbf{g}_h^{even}, \mathbf{g}_h^{odd} \in Y_{\alpha}$. Use the Bukhgeim-Cauchy Integral formula (13) to define the \mathcal{L} -analytic maps

$$(86) \quad \mathbf{v}^{even}(z) = \langle v_{-2}(z), v_{-4}(z), \dots \rangle := \mathcal{B}\mathbf{g}_h^{even}(z), \quad z \in \Omega,$$

$$(87) \quad \mathbf{v}^{odd}(z) = \langle v_{-3}(z), v_{-5}(z), \dots \rangle := \mathcal{B}\mathbf{g}_h^{odd}(z), \quad z \in \Omega.$$

By intertwining let also define

$$\mathbf{v}(z) := \langle v_{-2}(z), v_{-3}(z), \dots \rangle, \quad z \in \Omega.$$

By Proposition 2.1

$$(88) \quad \mathbf{v}^{even}, \mathbf{v}^{odd}, \mathbf{v} \in C^{1,\alpha}(\Omega; l_1) \cap C^{\alpha}(\bar{\Omega}; l_1) \cap C^2(\Omega; l_{\infty}).$$

Moreover, since $\mathbf{g}_h^{even}, \mathbf{g}_h^{odd}$ satisfy the hypothesis (83), by Theorem 2.2 we have

$$\mathbf{v}^{even}|_{\Gamma} = \mathbf{g}_h^{even} \quad \text{and} \quad \mathbf{v}^{odd}|_{\Gamma} = \mathbf{g}_h^{odd}.$$

In particular

$$(89) \quad v_n|_{\Gamma} = \sum_{k=0}^{\infty} (\alpha_k|_{\Gamma}) g_{n-k}, \quad n \leq -2.$$

For each $n \leq -2$, we use the convolution formula below to construct

$$(90) \quad u_n := \sum_{j=0}^{\infty} \beta_j v_{n-j}.$$

Since $a \in C^{2,\alpha}(\overline{\Omega})$, by (62), the sequence $z \mapsto \langle \beta_0(z), \beta_1(z), \beta_2(z), \dots \rangle$ is in $C^{2,\alpha}(\Omega; l_1) \cap C^{\alpha}(\overline{\Omega}; l_1)$. Since convolution preserves l_1 , the map is in

$$(91) \quad z \mapsto \langle u_{-2}(z), u_{-3}(z), \dots \rangle \in C^{1,\alpha}(\Omega; l_1) \cap C^{\alpha}(\overline{\Omega}; l_1).$$

Moreover, since $\mathbf{v} \in C^2(\Omega; l_{\infty})$ as in (88), we also conclude from convolution that

$$(92) \quad z \mapsto \langle u_{-2}(z), u_{-3}(z), \dots \rangle \in C^2(\Omega; l_{\infty}).$$

The property (91) justifies the calculation of traces $u_n|_{\Gamma}$ for each $n \leq -2$:

$$u_n|_{\Gamma} = \sum_{j=0}^{\infty} \beta_j|_{\Gamma} (v_{n-j}|_{\Gamma}).$$

Using (89) in the above equation gives

$$u_n|_{\Gamma} = \sum_{j=0}^{\infty} \beta_j|_{\Gamma} \sum_{k=0}^{\infty} \alpha_k|_{\Gamma} g_{n-j-k}.$$

A change of index $m = j + k$, simplifies the above equation

$$\begin{aligned} u_n|_{\Gamma} &= \sum_{m=0}^{\infty} \sum_{k=0}^m \alpha_k \beta_{m-k} g_{n-m}, \\ &= \alpha_0 \beta_0 g_n + \sum_{m=1}^{\infty} \sum_{k=0}^m \alpha_k \beta_{m-k} g_{n-m}. \end{aligned}$$

Using Lemma 4.1 (vi) yields

$$(93) \quad u_n|_{\Gamma} = g_n, \quad n \leq -2.$$

From the Lemma 4.2, the constructed u_n in (90) satisfy

$$(94) \quad \bar{\partial} u_n + \partial u_{n-2} + a u_{n-1} = 0, \quad n \leq -2.$$

Step 2: The construction of positive modes u_n for $n \geq 2$.

All of the positive Fourier modes are constructed by conjugation:

$$(95) \quad u_n := \overline{u_{-n}}, \quad n \geq 2.$$

Moreover using (93), the traces $u_n|_T$ for each $n \geq 2$:

$$(96) \quad u_n|_T = \overline{u_{-n}}|_T = \overline{g_{-n}} = g_n, \quad n \geq 2.$$

By conjugating (94) we note that the positive Fourier modes also satisfy

$$(97) \quad \bar{\partial}u_{n+2} + \partial u_n + au_{n+1} = 0, \quad n \geq 2.$$

Step 3: The construction of modes u_0, u_{-1} and u_1 .

Let $\psi \in \Psi_g^a$ as in (82) and define

$$(98) \quad u_0 := \psi,$$

and

$$(99) \quad u_{-1} := \frac{-\bar{\partial}\psi - \partial u_{-2}}{a}, \quad u_1 := \overline{u_{-1}}.$$

By the construction $u_0 \in C^2(\Omega; l_\infty)$ and $u_{-1} \in C^1(\Omega; l_\infty)$, and

$$(100) \quad \bar{\partial}u_0 + \partial u_{-2} + au_{-1} = 0$$

is satisfied. Furthermore, by conjugating (100) yields

$$(101) \quad \partial u_0 + \bar{\partial}u_2 + au_1 = 0.$$

Since $\psi \in \Psi_g^a$, the trace of u_0 satisfies

$$(102) \quad u_0|_T = g_0.$$

We check next that the trace of u_{-1} is g_{-1} :

$$\begin{aligned} u_{-1}|_T &= \left. \frac{-\bar{\partial}\psi - \partial u_{-2}}{a} \right|_T \\ &= -\frac{1}{a} \Big|_T \frac{e^{i\eta}}{2} (\partial_n + i\partial_\tau) \psi|_T - \frac{1}{a} \Big|_T \partial u_{-2}|_T \\ &= -\frac{1}{2a} \Big|_T e^{i\eta} \{ \partial_n \psi|_T + i\partial_\tau \psi|_T + 2e^{-i\eta} \partial u_{-2}|_T \} \\ (103) \quad &= g_{-1}, \end{aligned}$$

where the last equality uses (84) and the condition in class (82).

Step 4: The construction of the tensor field \mathbf{F}_ψ whose attenuated X-ray data is g .

We define the 2-tensor

$$(104) \quad \mathbf{F}_\psi := \begin{pmatrix} f_0 + 2\operatorname{Re} f_2 & 2\operatorname{Im} f_2 \\ 2\operatorname{Im} f_2 & f_0 - 2\operatorname{Re} f_2 \end{pmatrix},$$

where

$$(105) \quad f_0 = -2 \operatorname{Re} \left(\frac{\bar{\partial}\psi + \partial u_{-2}}{a} \right) + a\psi, \text{ and}$$

$$(106) \quad f_2 = -\bar{\partial} \left(\frac{\bar{\partial}\psi + \partial u_{-2}}{a} \right) + \partial u_{-3} + a u_{-2}.$$

Note that f_2 is well defined as $u_{-2} \in C^2(\Omega; l_\infty)$ from (92).

In order to show $g|_{\Gamma_+} = X_a \mathbf{F}_\psi$ with \mathbf{F}_ψ as in (104), we define the real valued function u via its Fourier modes

$$(107) \quad \begin{aligned} u(z, \theta) &:= u_0(z) + u_{-1}e^{-i\varphi} + \overline{u_{-1}}(z)e^{i\varphi} \\ &\quad + \sum_{n=2}^{\infty} u_{-n}(z)e^{-in\varphi} + \sum_{n=2}^{\infty} u_n(z)e^{in\varphi}. \end{aligned}$$

We check below that u is well defined, has the trace g on Γ and satisfies the transport equation (57).

For convenience consider the intertwining sequence

$$\mathbf{u}(z) := \langle u_0(z), u_{-1}(z), u_{-2}(z), u_{-3}(z), \dots \rangle, \quad z \in \Omega.$$

Since $\mathbf{u} \in C^{1,\alpha}(\Omega; l_1) \cap C^\alpha(\overline{\Omega}; l_1)$, by [30, Proposition 4.1 (iii)] we conclude that u is well defined by (107) and as a function in $C^{1,\alpha}(\Omega \times \mathbf{S}^1) \cap C^\alpha(\overline{\Omega} \times \mathbf{S}^1)$. In particular $u(\cdot, \theta)$ for $\theta = (\cos \varphi, \sin \varphi)$ extends to the boundary and its trace satisfies

$$\begin{aligned} u(\cdot, \theta)|_\Gamma &= \left(u_0 + u_{-1}e^{-i\varphi} + \overline{u_{-1}}e^{i\varphi} + \sum_{n=2}^{\infty} u_{-n}e^{-in\varphi} + \sum_{n=2}^{\infty} u_n e^{in\varphi} \right) \Big|_\Gamma \\ &= u_0|_\Gamma + u_{-1}|_\Gamma e^{-i\varphi} + \overline{u_{-1}}|_\Gamma e^{i\varphi} + \sum_{n=2}^{\infty} (u_{-n}|_\Gamma) e^{-in\varphi} + \sum_{n=2}^{\infty} (u_n|_\Gamma) e^{in\varphi} \\ &= g_0 + g_{-1}e^{-i\varphi} + g_1e^{i\varphi} + \sum_{n=2}^{\infty} g_{-n}e^{-in\varphi} + \sum_{n=2}^{\infty} g_n e^{in\varphi} \\ &= g(\cdot, \theta), \end{aligned}$$

where in the third equality we have used (93), (96), (102), and (103).

Since $u \in C^{1,\alpha}(\Omega \times \mathbf{S}^1) \cap C^\alpha(\bar{\Omega} \times \mathbf{S}^1)$, the following calculation is also justified:

$$\begin{aligned} \theta \cdot \nabla u + au &= e^{-i\varphi} \bar{\partial} u_0 + e^{i\varphi} \partial u_0 + e^{-2i\varphi} \bar{\partial} u_{-1} + \bar{\partial} u_1 + \partial u_{-1} + e^{2i\varphi} \partial u_1 \\ &\quad + \sum_{n=2}^{\infty} \bar{\partial} u_{-n} e^{-i(n+1)\varphi} + \sum_{n=2}^{\infty} \partial u_{-n} e^{-i(n-1)\varphi} \\ &\quad + \sum_{n=2}^{\infty} \bar{\partial} u_n e^{i(n-1)\varphi} + \sum_{n=2}^{\infty} \partial u_n e^{i(n+1)\varphi} \\ &\quad + au_0 + au_{-1} e^{-i\varphi} + au_1 e^{i\varphi} + \sum_{n=2}^{\infty} au_{-n} e^{-in\varphi} + \sum_{n=2}^{\infty} au_n e^{in\varphi}. \end{aligned}$$

Rearranging the modes in the above equation yields

$$\begin{aligned} \theta \cdot \nabla u + au &= e^{-2i\varphi} (\bar{\partial} u_{-1} + \partial u_{-3} + au_{-2}) + e^{2i\varphi} (\partial u_1 + \bar{\partial} u_3 + au_2) \\ &\quad + e^{-i\varphi} (\bar{\partial} u_0 + \partial u_{-2} + au_{-1}) + e^{i\varphi} (\partial u_0 + \bar{\partial} u_2 + au_1) \\ &\quad + \bar{\partial} u_1 + \partial u_{-1} + au_0 + \sum_{n=2}^{\infty} (\bar{\partial} u_{n+2} + \partial u_n + au_{n+1}) e^{i(n+1)\varphi} \\ &\quad + \sum_{n=2}^{\infty} (\bar{\partial} u_{-n} + \partial u_{-n-2} + au_{-n-1}) e^{-i(n+1)\varphi}. \end{aligned}$$

Using (94), (97), (100) and (101) simplifies the above equation

$$\begin{aligned} \theta \cdot \nabla u + au &= e^{-2i\varphi} (\bar{\partial} u_{-1} + \partial u_{-3} + au_{-2}) + e^{2i\varphi} (\partial u_1 + \bar{\partial} u_3 + au_2) \\ &\quad + \bar{\partial} u_1 + \partial u_{-1} + au_0. \end{aligned}$$

Now using (105) and (106), we conclude (57)

$$\theta \cdot \nabla u + au = e^{-2i\varphi} f_2 + e^{2i\varphi} \bar{f}_2 + f_0 = \langle \mathbf{F}_\psi \theta, \theta \rangle.$$

□

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